Exercise 333

For the following problems, consider the population of Ocean City, New Jersey, which is cyclical by season.

The population can be modeled by $P(t) = 82.5 - 67.5\cos[(\pi/6)t]$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). During a year, in what intervals is the population less than 20,000? During what intervals is the population more than 140,000?

Solution

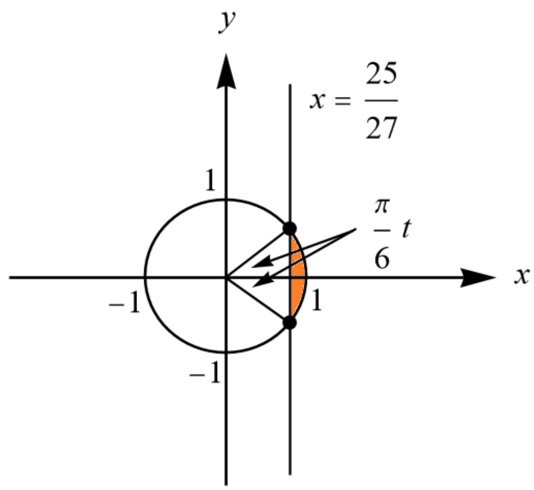
Set $P(t) < 20$ and solve the equation for t.

$$P(t) = 82.5 - 67.5\cos\left(\frac{\pi}{6}t\right) < 20$$

$$-67.5\cos\left(\frac{\pi}{6}t\right) < -62.5$$

$$\cos\left(\frac{\pi}{6}t\right) > \frac{25}{27}$$

Recall that the cosine of an angle measures the horizontal distance to points on the unit circle.



Taking the inverse cosine of 25/27 gives the counterclockwise angle from the positive x-axis. The clockwise angle is the same but negative because it's in the lower half of the unit circle.

$$-\cos^{-1}\left(\frac{25}{27}\right) < \frac{\pi}{6}t < \cos^{-1}\left(\frac{25}{27}\right)$$

Solve for t.

$$-\frac{6}{\pi}\cos^{-1}\left(\frac{25}{27}\right) < t < \frac{6}{\pi}\cos^{-1}\left(\frac{25}{27}\right)$$

$$-0.740 \lesssim t \lesssim 0.740$$

Since t is in months, multiply it by 30 to get the number of days: -22.2 days $\lesssim 30t \lesssim 22.2$ days. This means that between December 8 and January 23 the population is under 20,000.

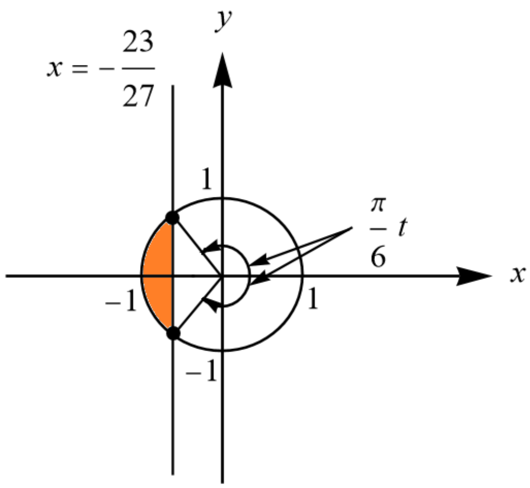
Now set $P(t) > 140$ and solve the equation for t.

$$P(t) = 82.5 - 67.5\cos\left(\frac{\pi}{6}t\right) > 140$$

$$-67.5\cos\left(\frac{\pi}{6}t\right) > 57.5$$

$$\cos\left(\frac{\pi}{6}t\right) < -\frac{23}{27}$$

Recall that the cosine of an angle measures the horizontal distance to points on the unit circle.



Taking the inverse cosine of $-23/27$ gives the counterclockwise angle from the positive x-axis. The clockwise angle is the same but negative because it's in the lower half of the unit circle; add 2π to it to make it positive.

$$\cos^{-1}\left(-\frac{23}{27}\right) < \frac{\pi}{6}t < -\cos^{-1}\left(-\frac{23}{27}\right) + 2\pi$$

Solve for t.

$$\frac{6}{\pi}\cos^{-1}\left(-\frac{23}{27}\right) < t < \frac{6}{\pi}\left[2\pi - \cos^{-1}\left(-\frac{23}{27}\right)\right]$$

$$4.95 \lesssim t \lesssim 7.05$$

Since t is in months, multiply it by 30 to get the number of days: 148.4 days $\lesssim 30t \lesssim$ 211.6 days. This means that between May 29 and August 2 the population is over 140,000.
(Note that 120 days is the start of May 1, and 210 days is the start of August 1.)

A plot of $P(t)$ versus t over the 12 months is shown here, verifying the results above.